DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **16**, Number **2**, October **2006**

Website: http://aimSciences.org

pp. **393–394**

A PIECE-WISE AFFINE CONTRACTING MAP WITH POSITIVE ENTROPY

Boris Kruglikov

Institute of Mathematics and Statistics
University of Tromsø, N-9037 Tromsø, Norway

Martin Rypdal

Institute of Mathematics and Statistics
University of Tromsø, N-9037 Tromsø, Norway

Abstract. We construct the simplest chaotic system with a two-point attractor on the plane.

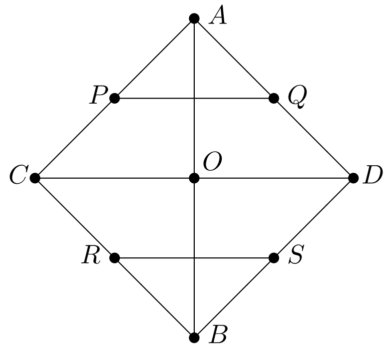
If $f : X \to X$ is an isometry of the metric space (X, d), then the topological entropy vanishes: $h_{\text{top}}(f) = 0$ (for definitions and notations, consult e.g. [4]).

This follows from the fact that the iterated distance $d_n^f = \max_{0 \le i < n} (f^i)^*(d)$ equals d. If f is distance non-increasing, the same equality holds, and again $h_{\text{top}}(f) = 0$.

Whenever f can have discontinuities of some tame nature so that f is piece-wise continuous, even the isometry result becomes difficult. In dimension two for invertible maps, it was proven by Gutkin and Haydn [3]. In arbitrary dimension, Buzzi proved that piece-wise affine isometries have zero topological entropy [2].

In the same paper after the theorem (remark 4), it is claimed that the result holds for arbitrary piece-wise (non-strictly) contracting maps. This latter is wrong, however, and the goal of this note is to present a counter-example.

Example: Let X be a rhombus $ADBC$ with vertices $(\pm 1, 0), (0, \pm 1)$; see the figure below. Let O be its center and P, Q, R, S be on the sides as is shown.



2000 *Mathematics Subject Classification.* Primary: 37B40, 37D50; Secondary: 37D25.
Key words and phrases. Piecewise affine maps, topological entropy.

Let f be partially defined on the rhombus, namely let it be defined on the interior of four big triangles forming the rhombus. These triangles are bijectively mapped by f as follows:

$$ACO \longrightarrow APQ, \quad ADO \longrightarrow BRS, \quad BCO \longrightarrow AQP, \quad BDO \longrightarrow BSR.$$

Thus the piece-wise affine map is defined.

If P, Q and R, S are middle-points of the intervals AC, AD and BC, BD, then the map is not strictly contracting. But if they are closer to the vertices A and B respectively than to C and D, then f is strictly contracting. In any case, the attractor of the system is the two-point set $\{A, B\}$. Notice that the points belong to the singularity set, where the map f is not (uniquely) defined.

Taking $\varepsilon = \frac{1}{2}$, we observe that the cardinality of minimal (n, ε)-spanning set satisfies: $2^{n+2} \le N(f, n, \varepsilon) \le 2^{n+3}$. In fact, if we partition CD into 2^n equal intervals $Z_i Z_{i+1}$, then every d_n^f ε-ball is contained in some triangle $AZ_i Z_{i+1}$ or $BZ_i Z_{i+1}$, and every such a triangle is covered by two d_n^f ε-balls.

Therefore the topological entropy $h_{\mathrm{top}}(f) = \log 2$ is positive. In addition, the Lyapunov spectrum is strictly negative at each point (for strict contractions), and no invariant measure exists, so the variational principle breaks.

The result of Buzzi [2] generalizes, however, in the following fashion:

Theorem. *Let f be a piece-wise affine map with restriction to each continuity component being conformal (non-strict) contraction. Then $h_{\mathrm{top}}(f) = 0$.*

Now we can repeat Buzzi's remark 4 [2]: the proof of his theorem 3 applies almost literally to the above case of piece-wise affine conformal contracting maps. Therefore we omit the proof.

Remark. *It is obvious that if the attractor consists of one point only, then $h_{\mathrm{top}}(f) = 0$. If the phase space $X \subset \mathbb{R}^1$ is one-dimensional and the map is (non-strictly) contracting, then again $h_{\mathrm{top}}(f) = 0$. We don't even need to require piece-wise affine property. This follows from the Buzzi proposition 4* [1]*, yielding $h_{\mathrm{top}}(f) \le h_{\mathrm{mult}}(f)$, where $h_{\mathrm{mult}}(f)$ is the multiplicity entropy because the latter always vanishes in dimension one.*

Thus, our example with two-points attractor and two-dimensional phase-space X is the simplest possible example of a contracting chaotic system.

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Final version received March 2006.

E-mail address: Boris.Kruglikov@matnat.uit.no; Martin.Rypdal@matnat.uit.no